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Classical Trajectories in Rindler Space

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Abstract: The nature of single particle classical phase space trajectories in Rindler space with non-hermitian PT-symmetric Hamiltonian have been studied both in the relativistic as well as in the non-relativistic scenarios. It has been shown that in the relativistic scenario, both positional coordinates and the corresponding canonical momenta are real in nature and diverges with time. Whereas the phase space trajectories are a set of hyperbolas in Rindler space. On the other hand in the non-relativistic approximation the spatial coordinates are complex in nature, whereas the corresponding canonical momenta of the particle are purely imaginary. In this case the phase space trajectories are quite simple in nature. But the spatial coordinates are restricted in the negative region only.

Keywords: Rindler space; Uniformly accelerated frame; Classical trajectory; Poisson's equation

1. Introduction

Exactly like the Lorentz transformations of space time coordinates in the inertial frame [1,2], the Rindler coordinate transformations are for the uniformly accelerated frame of reference with respect to some inertial one [3–9]. From the references [3–9], it can very easily be shown that the Rindler coordinate transformations are given by

$$ct = \left(\frac{c^2}{\alpha} + x'\right) \sinh\left(\frac{\alpha t'}{c}\right) \quad \text{and}$$
$$x = \left(\frac{c^2}{\alpha} + x'\right) \cosh\left(\frac{\alpha t'}{c}\right). \tag{1}$$

Hence it is a matter of simple algebra to prove that the inverse transformations are given by

$$ct' = \frac{c^2}{2\alpha} \ln\left(\frac{x+ct}{x-ct}\right) \quad \text{and} \quad x' = (x^2 - (ct)^2)^{1/2} - \frac{c^2}{\alpha}. \tag{2}$$

Here α indicates the uniform acceleration of the frame. Hence it can very easily be shown from Eqs. (1) and (2) that the square of the four-line element changes from

$$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 \quad \text{to}$$
$$ds^2 = \left(1 + \frac{\alpha x'}{c^2}\right)^2 d(ct')^2 - dx'^2 - dy'^2 - dz'^2, \tag{3}$$

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where the former line element is in the Minkowski space.

Hence the metric in the Rindler space can be written as

$$g^{\mu\nu} = \mathrm{diag}\left(\left(1+\frac{\alpha x}{c^2}\right)^2, -1, -1, -1\right), \tag{4}$$

whereas in the Minkowski space-time we have the usual form

$$g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{5}$$

It is therefore quite obvious that the Rindler space is also flat. The only difference from the Minkowski space is that the frame of the observer is moving with uniform acceleration. It has been noticed from the literature survey, that the principle of equivalence plays an important role in obtaining the Rindler coordinates in the uniformly accelerated frame of reference. According to this principle, an accelerated frame in absence of gravity is equivalent to a frame at rest in presence of a gravity. Therefore in the present scenario, α may be treated to be the strength of constant gravitational field for a frame at rest.

Now from the relativistic dynamics of special theory of relativity [1], the action integral is given by

$$S = -\alpha_0 c \int_a^b ds \equiv \int_a^b L dt, \tag{6}$$

where $\alpha_0 = -m_0 c$ [1] and m_0 is the rest mass of the particle and c is the speed of light in vacuum.

The Lagrangian of the particle may be written as

$$L = -m_0 c^2 \left[\left(1+\frac{\alpha x}{c^2}\right)^2 - \frac{v^2}{c^2}\right]^{1/2}, \tag{7}$$

where $\mathbf{v}$ is the three velocity vector. Hence the three momentum of the particle is given by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}, \quad \text{or} \tag{8}$$

$$\mathbf{p} = \frac{m_0 \mathbf{v}}{\left[\left(1+\frac{\alpha x}{c^2}\right)^2 - \frac{v^2}{c^2}\right]^{1/2}}. \tag{9}$$

Then from the definition, the Hamiltonian of the particle may be written as

$$H = \mathbf{p}.\mathbf{v} - L \quad \text{or} \tag{10}$$

$$H = m_0 c^2 \left(1+\frac{\alpha x}{c^2}\right)\left(1+\frac{p^2}{m_0^2 c^2}\right)^{1/2}. \tag{11}$$

Hence it can very easily be shown that in the non-relativistic approximation, the Hamiltonian is given by

$$H = \left(1+\frac{\alpha x}{c^2}\right)\left(\frac{p^2}{2m_0} + m_0 c^2\right). \tag{11a}$$

In the classical level, the quantities H, x and p are treated as dynamical variables. Further, it can very easily be verified that in the quantum mechanical scenario where these quantities are considered to be operators, the Hamiltonian H is not hermitian. However the energy eigen spectrum for the Schrödinger equation has been observed to be real [10]. This is found to be solely because of the fact that H is PT-invariant. Now it is well know that $PxP^{-1} = -x$, $PpP^{-1} = -p$, whereas $TpT^{-1} = -p$ and $P\alpha P^{-1} = -\alpha$ but $T\alpha T^{-1} = \alpha$. Therefore, it is a matter of simple algebra to show that $PT\ H\ (PT)^{-1} = H^{PT} = H$. As has been shown by several authors [11] that if H is PT-invariant, then the energy eigen values will be real. Here P and T are respectively the parity and the time reversal operators. Further if the Hamiltonian is PT symmetric, then H and PT should have common eigen states. In [10] we have noticed that the solution of the Schrödinger equation is obtained in terms of the variable $u = 1 + \alpha x/c^2$, which is PT-symmetric. Hence any function, e.g., Whittaker function $M_{k,\mu}(u)$ or Associated Laguerre function $L_m^n(u)$, the solution of the Schrödinger equation are PT-symmetric. These polynomials are also the eigen functions of the operator PT.

Of course with the replacement of hermiticity of the Hamiltonian with the PT-symmetry, we have not discarded the important quantum mechanical key features of the system described by this Hamiltonian and also kept the canonical quantization rule invariant, i.e., $TiT^{-1} = -i$. This point was also discussed in an elaborate manner in reference [11] and in some of the references cited there.

In this article we have investigated the time evolution for both the space and the momentum coordinates of the particle moving in Rindler space. We have considered both the relativistic and the non-relativistic form of the Rindler Hamiltonian (Eqs. (11) and (11a) respectively). Hence we shall also obtain the classical phase space trajectories for the particle in the Rindler space. We have noticed that in the relativistic scenario, both the spatial and the momentum coordinates are real in nature and diverge as $t \longrightarrow \infty$. For both the variables the time dependencies are extremely simple. Hence we have obtained classical trajectories $p(x)$ by eliminating the time dependent part.

However, in the non-relativistic approximation, the spatial coordinates are quite complex in nature, whereas the momentum coordinates are purely imaginary. Since the mathematical form of the phase space trajectories are quite complicated, we have obtained $p(x)$ numerically in the non-relativistic scenario.

In the first part of this article, we have considered the relativistic picture and obtained the phase space trajectories, whereas in the second part, the classical phase space structure is obtained for non-relativistic case. To the best of our knowledge such studies have not been done before.

2. Relativistic Picture

The classical Hamilton's equation of motion for the particle is given by [12]

$$\dot{x} = [H, x]_{p.x} \quad \text{and} \quad \dot{p} = [H, p]_{p,x} \tag{12}$$

where $[H, f]_{p,x}$ is the Poisson bracket and is defined by [12]

$$[f, g]_{p,x} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}. \tag{13}$$

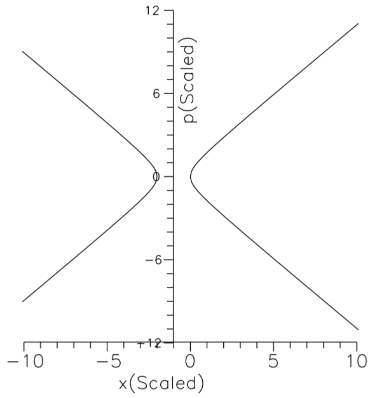

Fig. 1 Phase space trajectories for the relativistic scenario with the scaling parameters equal to unity

In this case $f = x$ or p. In Eq. (12) the dots indicate the derivative with respect to time. Now using the relativistic version of Rindler Hamiltonian from Eq. (11), the explicit form of the equations of motion are given by

$$\dot{x} = \left(1 + \frac{\alpha x}{c^2}\right) \frac{pc^2}{(p^2c^2 + m_0^2c^4)^{1/2}} \quad \text{and} \quad \dot{p} = -\frac{\alpha}{c}(p^2c^2 + m_0^2c^4)^{1/2}. \tag{14}$$

The parametric form of expressions for x and p represent the time evolution of spatial coordinate and the corresponding canonical momentum. The analytical expressions for time evolution for both the quantities can be obtained after integrating these coupled equations and are given by

$$x = \frac{c^2}{\alpha}[C_0 \cosh(\omega t - \phi) - 1] \quad \text{and} \quad p = -m_0 c \ \sinh(\omega t - \phi), \tag{15}$$

where C_0 and ϕ are the integration constants, which are real in nature and $\omega = \alpha/c$ is the frequency defined for some kind of quanta in [10]. Hence eliminating the time coordinate, we can write

$$\left(1 + \frac{\alpha x}{c^2}\right)^2 \frac{1}{C_0^2} - \frac{p^2}{m_0^2c^2} = 1. \tag{16}$$

This is the mathematical form of the set of classical trajectories of the particle in the phase space. Or in other wards, these set of hyperbolas are the classical trajectories of the particle in the Rindler space. This is consistent with the hyperbolic motion of the particle in a uniformly accelerated frame. These set of hyperbolic equations can also be written as

$$p^2 = m_0^2c^2 \left(\frac{2\alpha x}{c^2}\right)\left(1 + \frac{x\omega}{2c}\right). \tag{17}$$

It is quite obvious from the parametric form of the variation of x and p with time that both the quantities are unbound. This is also reflected from the nature of phase space trajectories as shown in Fig. 1 for the scaled x and p. The scaling factors are α/c^2 for x and $(m_0c)^{-1}$ for p. For the sake of illustration, we have chosen the arbitrary constant $C_0 = 1$.

In this figure we have also taken both the scaling factors identically equal to unity. Then obviously Eq. (16) reduces to

$$(x+1)^2 - p^2 = 1.$$

We shall get the other set of trajectories by choosing different values for the scaling factors. It is obvious that in this case the centre of the hyperbola is at $(-1, 0)$. Therefore with the increase of α, the centre $\longrightarrow (0, 0)$. Further the vertices for this particular hyperbolic curve are at $(0, 0)$ and $(-2, 0)$. The second one is in scaled form. Therefore for the gravitational field α large enough, both the vertices coincide at the centre $(0, 0)$. It is also obvious that for very large values of α, these two curves touch each other at $(0, 0)$. We have therefore noticed that the phase space trajectories are unbound and consistent with the motion of the particle in Rindler space.

3. Non-Relativistic Picture

We next consider the non-relativistic form of Rindler Hamiltonian given by Eq. (11a). Now following Eq. (12), the equations of motion for the particle in Rindler space in the non-relativistic approximation are given by

$$\dot{x} = \left(1 + \frac{\alpha x}{c^2}\right)\frac{p}{m_0} \quad \text{and} \quad \dot{p} = -\frac{\alpha}{c^2}\left(\frac{p^2}{2m_0} + m_0c^2\right). \tag{18}$$

On integrating the second one we have

$$p = i2^{1/2}m_0c\cot\left(\frac{2^{1/2}\omega t + \phi}{2}\right) = ip_I. \tag{19}$$

The particle momentum is therefore purely imaginary in nature with its real part $p_R = 0$. Here ϕ is a real constant phase. Next evaluating the first integral analytically, we have

$$x = \frac{c}{\omega}\left[-1 + \cos\left\{\ln\left(\sin^2\left(\frac{2^{1/2}\omega t - \phi}{2}\right)\right)\right\}\right] + i\frac{c}{\omega}\left[\sin\left\{\ln\left(\sin^2\left(\frac{2^{1/2}\omega t - \phi}{2}\right)\right)\right\}\right] = x_R + ix_I. \tag{20}$$

The spatial part is therefore complex in nature, where the real part

$$x_R = \frac{c}{\omega}\left[-1 + \cos\left\{\ln\left(\sin^2\left(\frac{2^{1/2}\omega t - \phi}{2}\right)\right)\right\}\right] \tag{21}$$

and the corresponding imaginary part is given by

$$x_I = \frac{c}{\omega}\left[\sin\left\{\ln\left(\sin^2\left(\frac{2^{1/2}\omega t - \phi}{2}\right)\right)\right\}\right]. \tag{22}$$

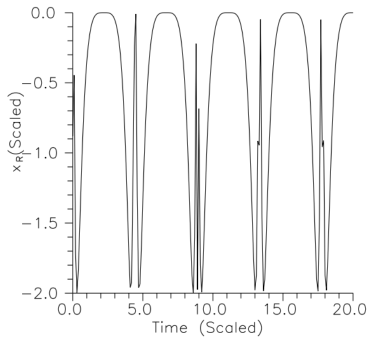

Fig. 2 Variation of scaled x_R with scaled time

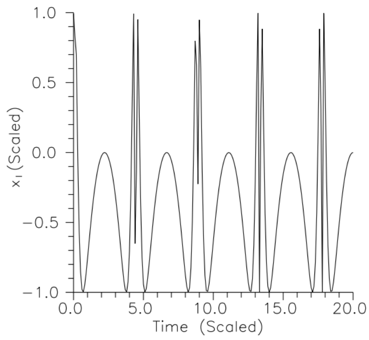

Fig. 3 Variation of scaled x_I with scaled time

Here again eliminating the time part, we have the mathematical form of phase space trajectories for the imaginary parts only

$$p_I = 2^{1/2} m_0 c \frac{\left[1 - \exp\left\{\sin^{-1}\left(\frac{\omega}{c} x_I\right)\right\}\right]^{1/2}}{\exp\left\{\frac{1}{2}\sin^{-1}\left(\frac{\omega}{c} x_I\right)\right\}}, \tag{23}$$

which gives the phase space trajectories of the particle in the Rindler space in non-relativistic scenario. It should be noted here that since the real part of the particle momentum is zero, we have considered the imaginary parts only. Since p_I is real, therefore $\mid \omega x_I/c \mid \leq 1$, i.e., can not have all possible values.

In Fig. 2 we have plotted the scaled x_R, i.e. $(\omega x_R/c)$ with scaled time $(\omega t/2^{1/2})$ for $\phi = 0$. Since the constant phase ϕ is completely arbitrary, for the sake of illustration we have chosen it to be zero. In this diagram the scaling factors are also taken to be unity. Now if we consider variation of the scaling factors, the qualitative nature of the graphs will not change but there will be quantitative changes.

In Fig. 3 we have plotted the scaled x_I, i.e., $(\omega x_I/c)$ with scaled time $(\omega t/2^{1/2})$ for $\phi = 0$. In this case also same type of changes as has been mentioned for x_R will be observed.

In Fig. 4 we have plotted the scaled p_I, which is actually $(p_I/2^{1/2}m_0c)$ with scaled time $(\omega t/2^{1/2})$ for $\phi = 0$. In this case also the scaling factors are exactly

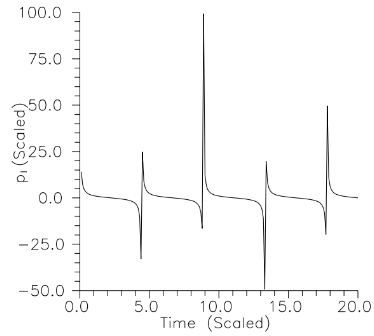

Fig. 4 Variation of scaled p_I with scaled time

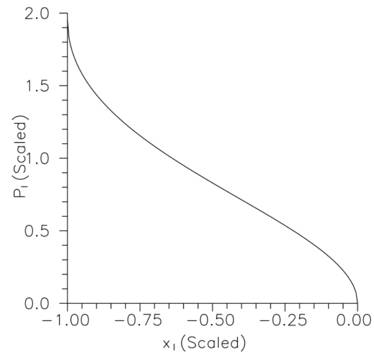

Fig. 5 Phase space trajectories for the non-relativistic scenario with the scaling parameters equal to unity

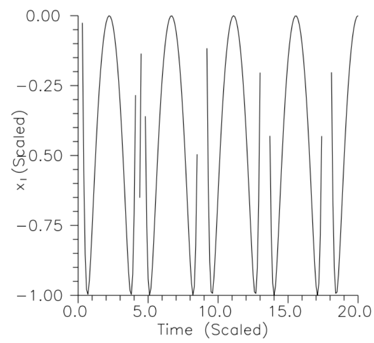

Fig. 6 Temporal variation of x_I in physically acceptable domain

equal to one. Further the same kind of variation as mentioned above will be observed for p_I with the change of scaling parameters.

Finally in Fig. 5 the phase space trajectory for scaled x_I and scaled p_I is shown Since the physically accepted domain for scaled x_I is from -1 to 0, we have shown in Figs. 6 and 7 the plot of scaled x_I and scaled p_I with scaled time.

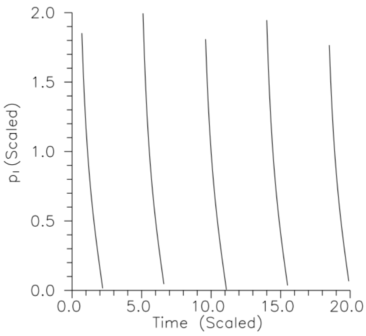

Fig. 7 Temporal variation of p_I in physically acceptable domain

4. Conclusion

Finally in conclusion we would like to mention that to the best of our knowledge this is the first time the phase space trajectories are obtained in Rindler space using non-hermitian PT-symmetric Hamiltonian.

In the relativistic case the trajectories can be represented by a set of hyperbolas. Whereas in the non-relativistic picture, particle momenta are purely imaginary and the space coordinates are complex in nature. The variation of real and imaginary parts of space coordinates are quite complicated. Further, the phase space is restricted within the domain of negative x-values. The imaginary part of particle momentum has been observed to change with time in a discrete manner in this region.

If we consider the Rindler Hamiltonian in the form

$$H = \left(1 + \frac{\alpha x}{c^2}\right) \frac{p^2}{2m} \tag{24}$$

then it is a matter of simple algebra to show that

$$1 + \frac{\omega x}{c} = t \exp(2m) \text{ and } p = \frac{2mc}{\omega t}. \tag{25}$$

Hence redefining $1+\omega x/c$ as new x and $2mc/(\omega p)\exp(2m)$ as new $1/p$, we have $xp = 1$, which gives the phase space trajectories in Rindler space. The trajectories are rectangular hyperbola.

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