



Slowly evolving noncommutative-geometry wormholes

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Abstract: This paper discusses noncommutative-geometry wormholes in the context of a cosmological model due to Sung-Won Kim. An ansatz suggested by the Friedmann-Lemaître-Robertson-Walker (FLRW) model leads to the assumption that the matter content can be divided into two parts, a cosmological part depending only on time and a wormhole part depending only on space. These assumptions are sufficient for deriving a complete zero-tidal force wormhole solution. The wormhole is evolving due to the scale factor in the FLRW model; it is restricted, however, to the curvature parameters $k = 0$ and $k = -1$. Unlike previous models, the noncommutative-geometry background affects both the wormhole part and the cosmological part of the solution.

Keywords: noncommutative geometry; wormholes; FLRW model

1 Introduction

Wormholes are handles or tunnels in spacetime connecting widely separated regions of our Universe or different universes altogether. While there had been some forerunners, macroscopic traversable wormholes were first discussed in detail by Morris and Thorne [1] in 1988. A few years later, Sung-Won Kim [2] proposed the possible existence of an evolving wormhole in the context of the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model by assuming that the matter content can be divided into two parts, the cosmological part that depends on time only and the wormhole part that depends on space only. The discussion was later expanded by Cataldo et al. [3].

The purpose of this paper is to study the relationship between wormholes inspired by noncommutative geometry and the Kim model. The noncommutative-geometry background simultaneously affects both the wormhole construction and the cosmological part of the solution. This result differs significantly from the outcomes in Refs. [2] and [3]. Regarding the strategy, Ref. [1] concentrates mainly on the wormhole geometry by specifying the metric coefficients. This strategy requires a search for matter or fields that can produce the energy-momentum tensor needed to sustain the wormhole. Here it needs to be emphasized that we are able to satisfy the geometric requirements from the physical properties. The result is an evolving zero-tidal force wormhole solution; it is restricted to the curvature parameters $k = 0$ and $k = -1$, corresponding to an open Universe.

Viewed from a broader perspective, it has already been shown that noncommutative geometry, which is an offshoot of string theory, can account for the flat galactic rotation curves [4,5], but under certain conditions, noncommutative geometry can also support traversable wormholes [6–11].

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This paper is organized as follows: Sec. 2 briefly recalls the structure of wormholes and the basic features of noncommutative geometry. Sec. 3 continues with the Sung-Won Kim model. Here the discussion is necessarily more detailed, partly in the interest of completeness, but mainly to allow the inclusion of a more general form of the Einstein field equations. These are subsequently used in Sec. 6 to obtain a wormhole solution that does not depend on the separation of the matter content. In Sec. 4 we derive a wormhole solution from the noncommutative-geometry background, followed by a discussion of the null energy condition in Sec. 5. Sec. 7 features a comparison to an earlier solution. In Sec. 8, we conclude.

2 Wormhole structure and noncommutative geometry

Morris and Thorne [1] proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

using units in which $c = G = 1$. Here $b = b(r)$ is called the *shape function* and $\Phi = \Phi(r)$ is called the *redshift function*, which must be everywhere finite to avoid an event horizon. For the shape function we must have $b(r_0) = r_0$, where $r = r_0$ is the radius of the *throat* of the wormhole. The wormhole spacetime should be asymptotically flat, i.e., $\lim_{r \rightarrow \infty} \Phi(r) = 0$ and $\lim_{r \rightarrow \infty} b(r)/r = 0$. An important requirement is the *flare-out condition* at the throat: $b'(r_0) < 1$, while $b(r) < r$ near the throat. The flare-out condition can only be met by violating the null energy condition (NEC), which states that

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0, \quad (2)$$

for all null vectors k^α , where $T_{\alpha\beta}$ is the energy-momentum tensor. Matter that violates the NEC is called “exotic” in Ref. [1]. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation has the form

$$T_{\alpha\beta} k^\alpha k^\beta = \rho + P^r < 0. \quad (3)$$

Here $T^t_t = -\rho$ is the energy density, $T^r_r = P^r$ is the radial pressure, and $T^\theta_\theta = T^\phi_\phi = P^t$ is the lateral pressure.

Returning now to the noncommutative-geometry background mentioned earlier, we need to recall that, as an offshoot of string theory, noncommutative geometry replaces point-like particles by smeared objects. (For a detailed discussion, see Refs. [12–14].) As a result, spacetime can be encoded in the commutator $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck’s constant discretizes phase space [13]. An interesting and effective way to model the smearing effect, discussed in Refs. [9, 15, 16], is to assume that the energy density of the static, spherically symmetric, smeared, and particle-like gravitational source is given by

$$\rho(r) = \frac{\mu\sqrt{\beta}}{\pi^2(r^2 + \beta)^2}, \quad (4)$$

which can be interpreted to mean that the gravitational source causes the mass μ of a particle to be diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty; so $\sqrt{\beta}$ has units of length. (Ref. [13] uses a Gaussian distribution instead of Eq. (4) to represent ρ .) Eq. (4) leads to the mass distribution

$$\int_0^r 4\pi(r')^2 \rho(r') dr' = \frac{2M\sqrt{\beta}}{\pi} \left(\frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} \right), \quad (5)$$

where M is now the total mass of the source.

According to Ref. [13], noncommutative geometry is one of the basic properties of spacetime that does not depend on particular features such as curvature. Moreover, since the noncommutative effects can be implemented by modifying only the energy-momentum tensor, there is no need to change the Einstein tensor in the field equations. As a result, the length scales need not be microscopic.

3 The Sung-Won Kim model

The Sung-Won Kim cosmological model with a traversable wormhole is given by [2]

$$ds^2 = -e^{2\Phi(r)} dt^2 + [R(t)]^2 \left[\frac{dr^2}{1 - kr^2 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (6)$$

where $R(t)$ is the scale factor of the Universe and k is the sign of the curvature of spacetime, i.e., $k = +1, 0$, or -1 . (So while $b = b(r)$ is still the shape function, it is subject to conditions that are different from those of a Morris-Thorne wormhole.) The Einstein field equations in Ref. [2] are based on Eq. (6), but it is subsequently assumed that $\Phi(r) \equiv 0$ to be consistent with the FLRW model.

The discussion of the Sung-Won Kim model is continued and elaborated on in Ref. [3]. Unfortunately, the field equations in Refs. [2] and [3] do not agree and need to be rederived. Here we follow Ref. [2] and base the calculations on line element (6). The results are

$$8\pi\rho(r, t) = 3 \left(\frac{\dot{R}}{R} \right)^2 e^{-2\Phi(r)} + \frac{3k}{R^2} + \frac{b'(r)}{R^2 r^2}, \quad (7)$$

$$8\pi P^r(r, t) = -2 \frac{\ddot{R}}{R} e^{-2\Phi(r)} - \left(\frac{\dot{R}}{R} \right)^2 e^{-2\Phi(r)} - \frac{k}{R^2} - \frac{b(r)}{R^2 r^3} + \frac{2}{R^2 r} \Phi'(r) \left(1 - kr^2 - \frac{b(r)}{r} \right), \quad (8)$$

$$\begin{aligned} 8\pi P^t(r, t) = & -2 \frac{\ddot{R}}{R} e^{-2\Phi(r)} - \left(\frac{\dot{R}}{R} \right)^2 e^{-2\Phi(r)} - \frac{k}{R^2} + \frac{b(r) - rb'(r)}{2R^2 r^3} \\ & + \frac{1}{R^2} \left[(\Phi'(r))^2 \left(1 - kr^2 - \frac{b(r)}{r} \right) + \Phi''(r) \left(1 - kr^2 - \frac{b(r)}{r} \right) \right. \\ & \left. - \frac{1}{2} \Phi'(r) \left(2kr + \frac{rb'(r) - b(r)}{r^2} \right) \right] + \frac{1}{R^2 r} \Phi'(r) \left(1 - kr^2 - \frac{b(r)}{r} \right), \end{aligned} \quad (9)$$

$$8\pi T_{01} = \frac{\dot{R}}{R^2} e^{-\Phi(r)} \Phi'(r) \left(1 - kr^2 - \frac{b(r)}{r} \right)^{1/2}, \quad (10)$$

where T_{01} is the outward energy flow. (The prime and overdots denote the derivatives with respect to r and t , respectively.)

If $\Phi(r) \equiv 0$, the results agree with those in Ref. [3] (omitting Λ , the cosmological constant). For convenience, these will now be restated:

$$8\pi\rho(r, t) = 3 \left(\frac{\dot{R}}{R} \right)^2 + \frac{3k}{R^2} + \frac{b'}{R^2 r^2}, \quad (11)$$

$$8\pi P^r(r, t) = -2 \frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R} \right)^2 - \frac{k}{R^2} - \frac{b}{R^2 r^3}, \quad (12)$$

$$8\pi P^t(r, t) = -2 \frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R} \right)^2 - \frac{k}{R^2} + \frac{b - rb'}{2R^2 r^3}. \quad (13)$$

The next step depends on a key insight, due to Sung-Won Kim [2], that allows the separation of the Einstein field equations into two parts, namely, the following *ansatz* for the matter parts:

$$R^2(t)\rho(r, t) = R^2(t)\rho_c(t) + \rho_w(r), \quad (14)$$

$$R^2(t)P^r(r, t) = R^2(t)P_c(t) + P_w^r(r), \quad (15)$$

$$R^2(t)P^t(r, t) = R^2(t)P_c(t) + P_w^t(r). \quad (16)$$

The subscripts c and w refer, respectively, to the cosmological and wormhole parts. So P_c necessarily represents the isotropic pressure.

The *ansatz* now allows us to separate Eqs. (11)-(13) into two parts, the left side being a function of t and the right side a function of r , also carried out in Ref. [3]:

$$R^2 \left[8\pi\rho_c - 3 \left(\frac{\dot{R}}{R} \right)^2 - \frac{3k}{R^2} \right] = \frac{b'}{r^2} - 8\pi\rho_w = l, \quad (17)$$

$$R^2 \left[8\pi P_c + 2 \frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} \right] = -\frac{b}{r^3} - 8\pi P_w^r = m, \quad (18)$$

$$R^2 \left[8\pi P_c + 2 \frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} \right] = \frac{b - rb'}{2r^3} - 8\pi P_w^t = m, \quad (19)$$

where l and m are constants. The reason is that a function of t cannot be equal to a function of r for all t and r unless they are equal to some constant. In Eqs. (18) and (19), the constants are the same since the cosmological parts are equal. The constants l and m may be taken as arbitrary.

Returning now to Eqs. (7)-(10), recall that we obtained the more general form of the field equations by using line element (6), as suggested in Ref. [2]. It now becomes apparent, however, that the separation in Eqs. (17)-(19) cannot be carried out by means of Eqs. (14)-(16) unless we assume that $\Phi(r) \equiv 0$, which takes us back to the FLRW model.

4 The noncommutative wormhole

To obtain a wormhole solution, we will consider the special case $l = -3m$, following Ref. [3]. Eqs. (17)-(19) then yield

$$3 \left(\frac{\dot{R}}{R} \right)^2 + \frac{3k}{R^2} - \frac{3m}{R^2} = 8\pi\rho_c \quad (20)$$

and

$$-2 \frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R} \right)^2 - \frac{k}{R^2} + \frac{m}{R^2} = 8\pi P_c, \quad (21)$$

for the cosmological part, while the wormhole part is given by

$$\frac{b'}{r^2} - 8\pi\rho_w = -3m, \quad (22)$$

$$-\frac{b}{r^3} - 8\pi P_w^r = m, \quad (23)$$

$$\frac{b - rb'}{2r^3} - 8\pi P_w^t = m. \quad (24)$$

The wormhole solution can be obtained from Eqs. (22)-(24) by making use of Eq. (4). It remains to be seen what restrictions will be placed on the solutions due to the cosmological part and to the necessary violation of the NEC. But for now we have from Eq. (22) and Eq. (4) that

$$b'(r) = 8\pi \frac{\mu\sqrt{\beta}r^2}{\pi^2(r^2 + \beta)^2} - 3mr^2. \quad (25)$$

Integrating and using the condition $b(r_0) = r_0$, we obtain

$$\begin{aligned} b(r) &= \frac{4M\sqrt{\beta}}{\pi} \left(\frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} \right) - mr^3 \\ &\quad - \frac{4M\sqrt{\beta}}{\pi} \left(\frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r_0}{\sqrt{\beta}} - \frac{r_0}{r_0^2 + \beta} \right) + mr_0^3 + r_0, \end{aligned} \quad (26)$$

where M is the mass of the wormhole.

It now becomes apparent that the resulting spacetime is not asymptotically flat. The normal procedure is to cut off the wormhole material at some $r = a$ and then join the structure to an external Schwarzschild spacetime. We will see in the next section, however, that the need to violate the NEC ($\rho + P^r < 0$) requires a slight modification of the shape function, resulting in the required asymptotic flatness.

5 Violating the NEC

To check the violation of the NEC ($\rho + P^r < 0$) for the wormhole, we let $R \equiv$ constant and obtain from Eqs. (11) and (12)

$$b - rb' - 2kr^3 > 0. \quad (27)$$

At $r = r_0$, we therefore get

$$r_0 - r_0 \frac{8\pi\mu\sqrt{\beta}r_0^2}{\pi^2(r_0^2 + \beta)^2} + 3mr_0^3 - 2kr_0^3 > 0. \quad (28)$$

Since $\sqrt{\beta}$ is extremely small, we actually have

$$r_0 + (3m - 2k)r_0^3 \geq 0. \quad (29)$$

To check this condition, we need to return to Ref. [13] for some additional observations. The relationship between the radial pressure and energy density is given by

$$P^r = -\rho. \quad (30)$$

The reason is that the source is a self-gravitating droplet of anisotropic fluid of density ρ and the radial pressure is needed to prevent the collapse back to the matter point. In addition, the lateral pressure is given by

$$P^t = -\rho - \frac{r}{2} \frac{\partial \rho}{\partial r}. \quad (31)$$

Since the length scales can be macroscopic, we can retain Eq. (30) and then use Eq. (31) to write

$$P^t = -\rho - \frac{r}{2} \frac{\partial \rho}{\partial r} = P^r + \frac{2\mu r^2 \sqrt{\beta}}{\pi^2(r^2 + \beta)^3} \quad (32)$$

by Eq. (4).

So on larger scales, we have $P^r = P^t$. Since the pressure becomes isotropic, we can assume the equation of state to be $P_c = -\rho_c$. Substituting in Eqs. (20) and (21), we get

$$-2\frac{\ddot{R}}{R} + 2\left(\frac{\dot{R}}{R}\right)^2 + \frac{2k}{R^2} - \frac{2m}{R^2} = 0. \quad (33)$$

This equation can be rewritten as

$$3\frac{\ddot{R}}{R} - 3\left(\frac{\dot{R}}{R}\right)^2 = \frac{3k}{R^2} - \frac{3m}{R^2}. \quad (34)$$

Subtracting the Friedmann equations

$$3\frac{\ddot{R}}{R} = -4\pi(\rho_c + 3P_c)$$

and

$$3\left(\frac{\dot{R}}{R}\right)^2 = 8\pi\rho_c - \frac{3k}{R^2}$$

now yields

$$\frac{3k}{R^2} - \frac{3m}{R^2} = -4\pi(\rho_c + 3P_c) - 8\pi\rho_c + \frac{3k}{R^2}. \quad (35)$$

So if $P_c = -\rho_c$, we obtain

$$m = 0, \quad (36)$$

independently of k .

Applied to Eq. (29), the NEC is violated if $k = 0$ or $k = -1$. These conditions correspond to an open Universe.

To summarize, we employed basic physical principles to derive the following zero-tidal force solution:

$$\Phi(r) \equiv 0 \quad (37)$$

and (since $m = 0$)

$$b(r) = \frac{4M\sqrt{\beta}}{\pi} \left(\frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} \right) - \frac{4M\sqrt{\beta}}{\pi} \left(\frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r_0}{\sqrt{\beta}} - \frac{r_0}{r_0^2 + \beta} \right) + r_0. \quad (38)$$

The slowly evolving wormhole solution is restricted to the values $k = 0$ and $k = -1$ to ensure that the NEC is violated. The wormhole spacetime is asymptotically flat.

6 The special case $k = 0$

For completeness let us briefly consider a wormhole solution that does not depend on the separation of the Einstein field equations. We can combine Eqs. (11) and (12) to obtain

$$8\pi r^3 R^2 [\rho(r, t) + P^r(r, t)] = 2r^3 (\dot{R}^2 - R\ddot{R}) + 2r^3 k + r b'(r) - b(r). \quad (39)$$

If we now let $k = 0$, then Eq. (6) represents an evolving Morris-Thorne wormhole with the usual shape function $b = b(r)$. The NEC is violated at the throat $r = r_0$ for all t whenever

$$8\pi r_0^3 R^2 [\rho(r_0, t) - P^r(r_0, t)] = 2r_0^3 (\dot{R}^2 - R\ddot{R}) + r_0 b'(r_0) - b(r_0) < 0. \quad (40)$$

If the Universe is indeed accelerating, then the term $-R\ddot{R}$ eventually becomes dominant due to the ever-increasing R . So for sufficiently large R , the NEC is violated, thereby fulfilling a key requirement for the existence of wormholes. (Inflating Lorentzian wormholes are discussed in Ref. [17].)

Recalling that the radial tension τ is the negative of P^r , Inequality (40) can be written (since $b(r_0) = r_0$)

$$8\pi r_0^2 R^2 [\tau(r_0) - \rho(r_0)] = 2r_0^2 (-\dot{R}^2 + R\ddot{R}) - b'(r_0) + 1 > 0. \quad (41)$$

If $R(t) \equiv 1$, this reduces to the static Morris-Thorne wormhole; so if $b'(r_0) < 1$, then $\tau(r_0) > \rho(r_0)$, requiring exotic matter. In Inequality (41), however, $\tau(r_0) > \rho(r_0)$ could result from the dominant term $R\ddot{R}$. In that case, the NEC is violated without requiring exotic matter for the construction of the wormhole itself.

7 Comparison to an earlier solution

A wormhole solution inspired by noncommutative geometry had already been considered in Ref. [9]. The Einstein field equation $\rho(r) = b'(r)/(8\pi r^2)$, together with Eq. (4), leads directly to the static solution, Eq. (38). (Here it is understood that $k = 0$, but $R(t)$ could be retained.) Unfortunately, this simple approach leaves the redshift function undetermined. The desirability of zero tidal forces then suggested the assumption $\Phi(r) \equiv 0$ in Ref. [9]. It is shown in Ref. [18], however, that this assumption causes a Morris-Thorne wormhole to be incompatible with the Ford-Roman constraints from quantum field theory. Given the noncommutative-geometry background, rather than the purely classical setting in Ref. [1], this objection does not apply directly.

It is interesting to note that in the present paper, the zero-tidal force solution is built into the Sung-Won Kim model and does not require any additional considerations.

8 Conclusion

Morris-Thorne wormholes typically require a reverse strategy for their theoretical construction: specify the geometric requirements and then manufacture or search the Universe for matter or fields to obtain the required energy-momentum tensor. One of the goals in this paper is to obtain a complete wormhole solution from certain physical principles. To this end, we assume a noncommutative-geometry background, as in previous studies, but we also depend on a cosmological model due to Sung-Won Kim that is based on the FLRW model with a traversable wormhole. The basic assumption is that the matter content can be divided into two parts, a cosmological part that depends only on t and a wormhole part that depends only on the radial coordinate r . The result is a complete zero-tidal force solution; it is restricted, however, to the values $k = 0$ and $k = -1$, corresponding to an open Universe. This conclusion is consistent with the special case $k = 0$ discussed in Sections 6 and 7.

The wormhole is slowly evolving due to the scale factor $R(t)$ and, critically, the noncommutative-geometry background not only produces the wormhole solution, it also affects in a direct manner the cosmological part of the solution. This conclusion differs significantly from those in Refs. [2] and [3].

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